One Dimensional (1D) and Two-Dimensional (2D) Spring Mass Chains

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Wave Propagation in Linear and Nonlinear Periodic Media:
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Coarse approximation of a uniform rod:

- Rod is discretized into \( N \) elements of length \( a \);
- Mass and stiffness distributions are described as lumped parameters;

\[
\begin{align*}
    k &= \frac{EA}{a} \\
    m &= \rho Aa \\
    x_n &= na
\end{align*}
\]
• System under consideration is the first, simplest example of a PERIODIC structure:
  – Here obtained considering a “dumb” discretization of a continuous rod;
  – Can be considered as a simple academic exercise

  – System initially studied by Newton (1686) to calculate the speed of sound in air:
    • Newton, *Principia*, Book II, 1686.

  – System is used by John Bernoulli and son Daniel (1727) to demonstrate that a system of N masses is characterized by N modes of vibration and associated frequencies

  – Configuration considered by Baden-Powell (1841) to calculate the velocity of wave propagation along one axis of a cubic lattice structure

  – Results later corrected and expanded by Lord Kelvin (1881)

  – Detailed discussions can be found in:
System’s behavior is governed by \( N \) equations of the kind:

\[
m\dddot{u}_n + 2k\dot{u}_n - k(u_{n+1} + u_{n-1}) = 0 \quad n = 1, \ldots, N
\]

- Impose a harmonic solution

\[
u(x_n, t) = u_n(t) = u_n(\omega)e^{-j\omega t}
\]

- Impose a wave solution

\[
u_n(\omega) = u_0[\kappa(\omega)]e^{j\kappa x_n}
\]

\[
u_n(\omega) = u_0[\mu(\omega)]e^{j\mu n}
\]

where

\[
\mu(\omega) = \kappa(\omega)a
\]

- Under the assumption that no external forces are applied:
  - Free wave propagation
Dispersion relations

- Substitute wave solution at frequency $\omega$ in $n$-th equation:

\[
\left[(-\omega^2 m + 2k)e^{i\mu n} - k(e^{i\mu(n-1)} + e^{i\mu(n+1)})\right]u_0(\mu) = 0
\]

\[
\left[(-\omega^2 m + 2k) - k(e^{-i\mu} + e^{i\mu})\right]u_0(\mu) = 0
\]

\[
\left[(-\omega^2 m + 2k(1 - \cos \mu))\right]u_0(\mu) = 0
\]  

\[\neq 0 \quad \text{Non-trivial solutions}
\]

- Dispersion relation (frequency – wavenumber relations):

\[-\omega^2 m + 2k(1 - \cos \mu) = 0 \quad \Rightarrow \quad \omega_0^2 = \frac{k}{m}
\]

\[\Omega^2 = 2(1 - \cos \mu) \quad \Rightarrow \quad \Omega = \frac{\omega}{\omega_0}
\]
\[ \omega = \omega(\mu): \text{direct solution} \]

- **Continuous rod vs. discrete:**
  - Discretization process can be described in terms of \textit{FINITE DIFFERENCE} formalism
    \[ 2u_n - u_{n+1} - u_{n-1} \approx a^2 \frac{\partial^2 u(x)}{\partial x^2} \]
  - This approximation is used in deriving equivalent continuum systems for discrete assemblies
  - Discretization causes the system to be dispersive
    \[ \frac{\partial \omega}{\partial k} \neq \frac{\omega}{k} \quad c_g \neq c_p \]
• Dispersion relation is PERIODIC in the wavenumber space:

\[ \Omega(\mu) = \Omega(\mu + 2\pi p) \quad p \text{ integer} \]

\[ \omega(\kappa) = \omega(\kappa + p\frac{2\pi}{a}) \]

- \( k \)-space is periodic of period \( 2p/a \)

• As a result, displacements are also periodic in the wavenumber space

\[
\left[ -\omega^2 m + 2k(1 - \cos(\mu + 2\pi p)) \right] u_0(\mu + 2\pi p) = 0
\]

\[ \Rightarrow \left[ -\omega^2 m + 2k(1 - \cos \mu) \right] u_0(\mu) = 0 \]

\[ u_0(\mu + 2\pi p) = u_0(\mu) \]
• Result is due to the SAMPLING of a continuous system:

\[ u(x) \rightarrow u(na) \]

– Spatial sampling occurs at a frequency

\[ \kappa_s = \frac{2\pi}{a} \]

– The result

\[ u_0(\mu + 2\pi p) = u_0(\mu) \]

\[ u_0(\kappa + p\frac{2\pi}{a}) = u_0(\kappa) \]

is an expression of the Sampling Theorem (Shannon) theorem, for a system sampled in space

• A single period of the wavenumber/frequency relation for a periodic system is called:

FIRST BRILLOUIN ZONE
$\omega = \omega(\mu)$: direct solution

First Brillouin Zone

Irreducible Brillouin Zone
• Analogy with time-domain signals can be used to obtain a good guess about the NATURAL FREQUENCIES of a FINITE PERIODIC system with N masses (free-free for simplicity):
  – Finite system can be considered as a truncation of an infinite one
  – Truncation causes the system to be DISCRETE instead of continuous

\[ u_0(\kappa) \rightarrow u_0(p\Delta\kappa) \]

where wavenumber resolution is:

and

\[ \Delta\kappa = \frac{\pi}{Na} \]

\[ p = -N, -N + 1, \ldots, N - 1, N \]

– Discrete wavenumber values correspond to N values of frequencies
– Natural frequencies can be read directly on the dispersion curve, given the number of masses and boundary conditions
Natural frequencies ($N=5$)

Is parallel with finite time signal completely true?

Not quite....

A factor 2 is missing!!!!
\[ \mu = \mu(\omega): \text{inverse solution} \]

Alternatively, the solution of the dispersion relation:

\[ \Omega^2 = 2(1 - \cos \mu) \]

can be found by imposing frequency:

\[ \mu = \cos^{-1}(1 - \frac{\Omega^2}{2}) \]

where

\[ \mu \text{ is real for } \Omega \leq 2 \quad \mu \text{ is imaginary for } \Omega > 2 \]

The wave solution to the governing equation:

\[ m\ddot{u}_n + 2k u_n - k(u_{n+1} + u_{n-1}) = 0 \]

should be expressed as follows

\[ u_n(\omega) = u_0(\omega)e^{i\mu n} \]

\[ Im[\mu] \quad \text{Attenuation constant} \]
Harmonic response of a finite system
(N masses)
Harmonic response of a finite system (N=100 masses)

$\Omega = 0.1$

$\Omega = 0.5$

$\Omega = 1.5$

$\Omega = 1.96$
Harmonic response of a finite system
(N=100 masses)

\[ \Omega = 2.01 \]

\[ \Omega = 2.05 \]

\[ \Omega = 2.1 \]
Wave speeds

**Phase velocity:**

\[ c_{ph} = \frac{\omega}{\kappa} = a\omega_0 \frac{|\sin \mu/2|}{|\mu/2|} \]

**Group velocity:**

\[ c_g = \frac{\partial \omega}{\partial \kappa} = a \frac{\partial \omega}{\partial \mu} \]

\[ c_g = a\omega_0 \cos \frac{\mu}{2} \]
Average Energy

- **Average energy density**: sum of average potential and kinetic energy of the unit cell:
  - **Average potential energy**
    \[
    \langle U \rangle = \frac{U}{a} = \frac{1}{2a} k \text{Re}[(u_n - u_{n-1})^2] \\
    u_n - u_{n-1} = A \text{Re}[1 - e^{i\mu}] \\
    \langle U \rangle = \frac{1}{2a} A^2 k (1 - e^{i\mu})(1 + e^{-i\mu}) \\
    \langle U \rangle = A^2 \frac{k}{a} \sin^2 \frac{\mu}{2}
    \]
  - **Average kinetic energy**
    \[
    \langle K \rangle = \frac{K}{a} = \frac{1}{2a} m \text{Re}[\dot{u}_n^2] \\
    \langle K \rangle = \frac{m A^2}{4a} \omega^2 \\n    \dot{u}_n = \text{Re}[j\omega u_n] \\
    \langle K \rangle = A^2 \frac{k}{a} \sin^2 \frac{\mu}{2} = \langle U \rangle
    \]
  - **Total energy**
    \[
    \langle \mathcal{E} \rangle = \langle K \rangle + \langle U \rangle \\
    \langle \mathcal{E} \rangle = 2 A^2 \frac{k}{a} \sin^2 \frac{\mu}{2}
    \]
Energy flow

- Energy flow from one cell to the next is the AVERAGE POWER flowing from one cell to the next

\[ f_{n,n+1} = k(u_{n+1} - u_n) \]

\[ \Phi = - \langle \mathcal{R} e[f_{n,n+1}] \times \mathcal{R} e[u_n] \rangle \]

\[ \Phi = -kA^2\omega_0 \sin \frac{\mu}{2} \sin \mu \]

- Energy velocity: rate at which energy flows along the lattice

\[ c_{\mathcal{E}} = \frac{\Phi}{\langle \mathcal{E} \rangle} \quad c_{\mathcal{E}} = \frac{A^2 k\omega_0 \sin \mu \sin \frac{\mu}{2}}{2kA^2/\sin^2 \frac{\mu}{2}} \]

\[ c_{\mathcal{E}} = a\omega_0 \cos \frac{\mu}{2} \]

- The energy velocity equals the group velocity

\[ c_{\mathcal{E}} = c_g \]
System is representative of:

- Bi-material rod:

- NaCl crystal along one of lattice directions

- GaAs zincblende crystal: vibration of the (1 0 0) plane
• Governing equations for $2n$ and $(2n+1)$ masses:

$$m_b \quad \cdots \quad m_a$$

$$k \quad a \quad a$$

• Impose a solution of the kind:

$$-\omega^2 m_a u_{2n} + 2k u_{2n} - k(u_{2n+1} + u_{2n-1}) = 0$$

$$-\omega^2 m_b u_{2n+1} + 2k u_{2n+1} - k(u_{2n+2} + u_{2n}) = 0$$

$$u_{2n}(\omega) = u_a(\omega)e^{jkx_{2n}} = u_a(\omega)e^{j2akn}$$

$$u_{2n+1}(\omega) = u_b(\omega)e^{jkx_{2n+1}} = u_b(\omega)e^{j(2n+1)ak}$$

This solution describes waves propagating only through particles (a) and (b). Wavelength and frequencies are the same, but the amplitudes of the two waves are not equal.
Dispersion relations

- Substituting gives:

\[(2k - \omega^2 m_a)u_a - k(e^{j\kappa a} + e^{-j\kappa a})u_b = 0\]
\[(2k - \omega^2 m_b)u_b - k(e^{j\kappa a} + e^{-j\kappa a})u_a = 0\]

- In matrix form:

\[
\begin{pmatrix}
-\omega^2 m_a + 2k & -2k \cos \kappa a \\
-2k \cos \kappa a & -\omega^2 m_b + 2k
\end{pmatrix}
\begin{pmatrix}
u_a \\
u_b
\end{pmatrix} = 0
\]

- Characteristic equation

\[
\omega^4 - 2k\left(\frac{1}{m_a} + \frac{1}{m_b}\right)\omega^2 + 4\frac{k^2}{m_a m_b} \sin^2 \kappa a = 0
\]

- Solution identifies TWO BRANCHES:

\[
\omega = \omega_1(\kappa) \text{ and } \omega = \omega_2(\kappa)
\]
Dispersion relations

• where:

\[
\omega_1^2(\kappa) = k\left(\frac{1}{m_a} + \frac{1}{m_b}\right) - \sqrt{\left(\frac{1}{m_a} + \frac{1}{m_b}\right)^2 - \frac{4}{m_a m_b} \sin^2 \kappa a}
\]

ACOUSTIC BRANCH

\[
\omega_2^2(\kappa) = k\left(\frac{1}{m_a} + \frac{1}{m_b}\right) + \sqrt{\left(\frac{1}{m_a} + \frac{1}{m_b}\right)^2 - \frac{4}{m_a m_b} \sin^2 \kappa a}
\]

OPTICAL BRANCH

• Both branches are PERIODIC in the wavenumber domain:

\[
\kappa a \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]
\]
\[ \omega = \omega(\kappa) \] Direct solution

OPTICAL BRANCH

ACOUSTIC BRANCH

Band gap

\[ \sqrt{\frac{2k}{m_a}} \]

\[ \sqrt{\frac{2k}{m_b}} \]
\[ \omega = \omega(\kappa) \text{ Direct solution} \]

\[ m_a = m_b \]

Band-gap disappears
First Brillouin zone

• Period of wavenumber/frequency domain is:
  
  – Single mass system
    \[ \kappa a \in [-\pi, +\pi] \]
  
  – Two-mass system
    \[ \kappa a \in \left[-\frac{\pi}{2}, +\frac{\pi}{2}\right] \quad \rightarrow \quad 2\kappa a \in [-\pi, +\pi] \]

• The period of the dispersion relation is always given by:
  
  where:
  
  \[ \kappa d \in [-\pi, +\pi] \]

  – Single mass system:
  
    \[ d = a \]

  – Two mass system:
  
    \[ d = 2a \]
First Brillouin zone

• For any 1D periodic system, the frequency/wavenumber spectrum is periodic in the domain:

\[ \kappa d \in [-\pi, +\pi] \]

where \( d \) - Spatial period of the structure

• Propagation constant:

\[ \mu = \kappa d \]

• First Brillouin zone:

\[ \mu \in [-\pi, +\pi] \]

The definition of the Brillouin zone can be used to define unequivocally the SPATIAL PERIOD of the system
$\mu = \mu(\omega)$: inverse solution
Response of a system of $N=200$ masses

$\Omega = 0.1$

$\Omega = 1.3$
Response of a system of N=200 masses

\[ \Omega = 1.6 \]

\[ \Omega = 1.83 \]
Spring Mass System

Unit Cell:

\[ m \]

\[ k_x \]

\[ k_y \]
Governing equations & wave solution

• Equation of harmonic motion for mass $n,m$:

\[ -\omega^2 m u_{n,m} + (2k_x + 2k_y)u_{n,m} \ldots \]
\[ \ldots - k_x(u_{n+1,m} + u_{n-1,m}) - k_y(u_{n,m+1} + u_{n,m-1}) = 0 \]

• Wave propagation solution:

\[ u_{n,m}(\omega) = u_0\left[\kappa(\omega)\right]e^{j\kappa \cdot r} \]

– where

\[ \kappa = \kappa_x i + \kappa_y j \]
\[ r = xi + yj \]
\[ r = na_x i + ma_y j \]
• Direction of wave propagation

\[ \kappa = \kappa_x i + \kappa_y j \]

\[ \kappa = \kappa (\cos \theta i + \sin \theta j) \]

\[ \theta = \tan^{-1} \frac{k_y}{k_x} \]

\[ r = x i + y j \]

Wave front
Wave propagation solution

- Rewrite solution as:

\[ u_{n,m}(\omega) = u_0[\kappa(\omega)]e^{j\kappa \cdot r} \]
\[ = u_0[\kappa(\omega)]e^{j(\kappa_x a_x n + \kappa_y a_y m)} \]
\[ = u_0[\kappa(\omega)]e^{j(\mu_x n + \mu_y m)} \]

- and

\[ u_{n\pm1,m\pm1} = u_0[\kappa(\omega)]e^{j(\mu_x (n\pm1) + \mu_y (m\pm1))} \]
\[ u_{n\pm1,m\pm1} = u_{n,m}(\omega)e^{j(\pm\mu_x \pm \mu_y)} \]
Dispersion relation

• Substituting in governing equation leads to:

\[
[(-\omega^2 m + 2k_x + 2k_y) - k_x(e^{-j\mu_x} + e^{j\mu_x}) - k_y(e^{-j\mu_y} + e^{j\mu_y})]u_0(\mu_x, \mu_y) = 0
\]

\[
[-\omega^2 m + 2k_x(1 - \cos \mu_x) + 2k_y(1 - \cos \mu_y)]u_0(\mu_x, \mu_y) = 0
\]

• 2D dispersion relation

\[
-\omega^2 m + 2k_x(1 - \cos \mu_x) + 2k_y(1 - \cos \mu_y) = 0
\]

\[
\omega = \omega(\mu_x, \mu_y)
\]

Surface in the wavenumber domain
2D Dispersion relation

\[
\begin{align*}
\kappa_x &= \kappa_y = 1 \\
\mu &= 1
\end{align*}
\]
2D Dispersion relation

\[ k_x = k_y = 1 \]
\[ m = 1 \]
Group velocity

• According to definition:

\[ \mathbf{c}_g = \nabla \omega(\mu_x, \mu_y) \]

\[ \mathbf{c}_g = c_{g_x} \mathbf{i} + c_{g_y} \mathbf{j} \]

– Where

\[ c_{g_x} = \frac{\partial \omega}{\partial \kappa_x} = a_x \frac{\partial \omega}{\partial \mu_x} \]

\[ c_{g_y} = \frac{\partial \omega}{\partial \kappa_y} = a_y \frac{\partial \omega}{\partial \mu_y} \]

• Recall that:

– Velocity of energy flow equals the group velocity

– Energy flows in the direction corresponding to the group velocity
In this case:

\[
c_{g_x} = a_x \frac{\partial \omega}{\partial \mu_x} = \frac{a_x}{2} \frac{k_x/m \sin \mu_x}{[k_x/m(1-\cos \mu_x) + k_y/m(1-\cos \mu_y)]^{1/2}}
\]

\[
c_{g_y} = a_y \frac{\partial \omega}{\partial \mu_y} = \frac{a_y}{2} \frac{k_y/m \sin \mu_y}{[k_x/m(1-\cos \mu_x) + k_y/m(1-\cos \mu_y)]^{1/2}}
\]

\[
c_g = c_{g_x} \mathbf{i} + c_{g_y} \mathbf{j}
\]

\[
\frac{k_y \sin \mu_y}{k_x \sin \mu_x} = \tan \phi
\]

Assume

\[
a_y = a_x
\]
Contour of dispersion surface

• Contour at a single frequency

\[ \omega_0 = 1 \]

From dispersion relations:

\[ \frac{\partial \mu_y}{\partial \mu_x} = -\frac{a_x k_x \sin \mu_x}{a_y k_y \sin \mu_y} = \tan \psi \]

Direction of energy flow at a given frequency and direction is perpendicular to isofrequency contour
Dispersion surface vs. group velocity

\[ k_x = k_y = 1 \quad m = 1 \]
\[ \omega_0 = 0.1 \]
Dispersion surface vs. group velocity

\[ k_x = k_y = 1 \quad m = 1 \]
\[ \omega_0 = 0.5 \]
Dispersion surface vs. group velocity

\[ k_x = k_y = 1 \quad m = 1 \]
\[ \omega_0 = 1.2 \]
Dispersion surface vs. group velocity

\[ k_x = k_y = 1 \quad m = 1 \]

\[ \omega_0 = 1.4 \]
Dispersion surface vs. group velocity

\[ k_x = k_y = 1 \quad m = 1 \]

\[ \omega_0 = \sqrt{2} \]
• Propagation of waves is strongly directional at specified frequency
• At those frequencies, waves propagate only in certain directions

BEAMING PHENOMENA

• For considered configuration beaming is a very focused, but very narrow-band phenomenon
Example: 2D spring-mass lattice

Harmonic response of 40*40 lattice:

\[ \omega_0 = 0.5 \]

\[ \omega_0 = \sqrt{2} \]
Time domain response

Time domain simulations
Input modulated sine burst at various frequencies $w_e$
Time domain response

\[ \omega_0 = 0.5 \]

\[ \omega_0 = \sqrt{2} \]
Anisotropic lattice

\[ k_x \neq k_y \]

\[ k_x = 1, \ k_y = 1.2 \]
$k_x = 1, k_y = 1.5$

Anisotropic lattice

$k_x \neq k_y$
Anisotropic lattice

\[ k_x \neq k_y \]

\[ k_x = 1, k_y = 2 \]
• “Forbidden propagation” zone

\[ k_x = 1, \quad k_y = 2 \]
\[ \omega_0 = 1.6 \]

Waves do not propagate along the x direction
Example: 2D spring-mass lattice

Harmonic response of 40*40 lattice:

\[ k_x = 2, \ k_y = 1 \]